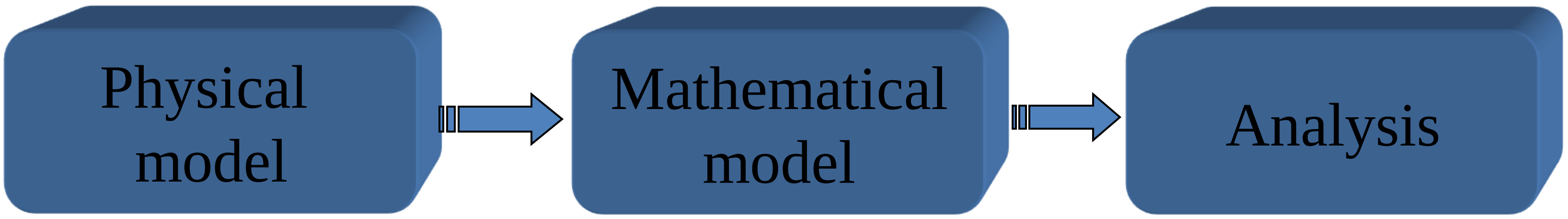
모드 해석

1. 모드해석이란

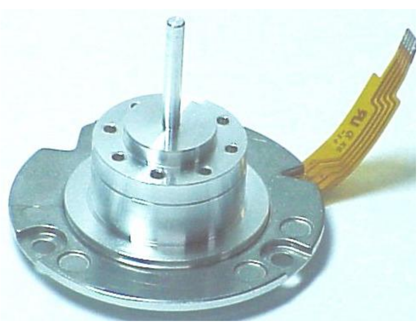


※ Magnetic field analysis

Physical model → Mathematical model → Analysis

Physical model

- simplification
- idealization

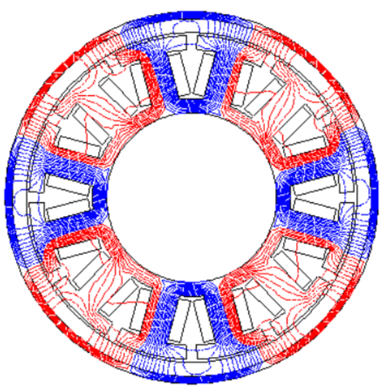


Mathematical model

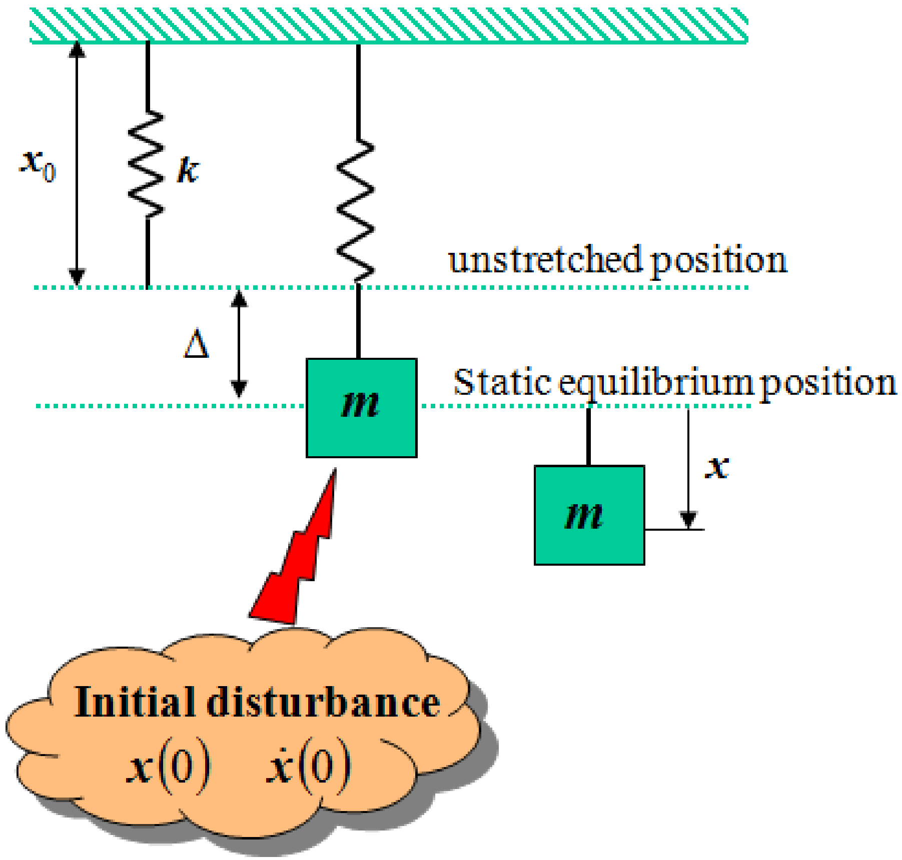
- governing equation
  Maxwell equation

Analysis

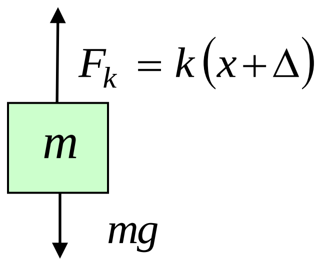
- analytical method
- numerical method



2. 1자유도 시스템의 비감쇠계 자유진동



< Free Body Diagram >



< Newton's 2nd law >

$$\sum \vec{F} = m\vec{a}$$

$$-k(x+\Delta) + mg = ma$$

$$m\ddot{x} + kx = 0$$

equation of motion

equation of motion

$$m\ddot{x} + kx = 0$$

initial conditions

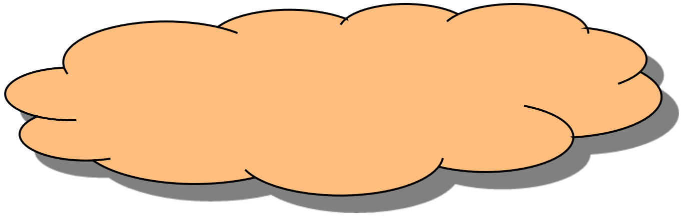
$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

solution

$$x(t) = A\sin(\omega_n t + \phi)$$

$$= \frac{\sqrt{\omega_n{}^2 x_0{}^2 + v_0{}^2}}{\omega_n}\sin(\omega_n t + \tan^{-1}\frac{\omega_n x_0}{v_0})$$

$$\omega_n = \sqrt{\frac{k}{m}} \quad [rad/s]$$



Displacement

$$x(t) = A\sin(\omega_n t + \phi)$$

Velocity

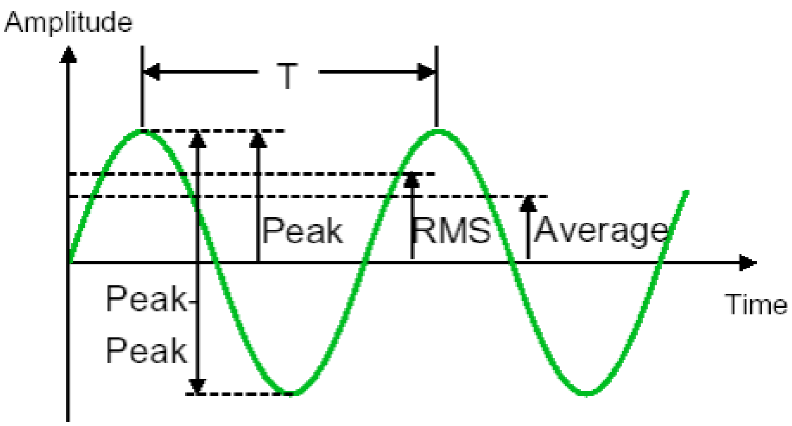
$$v(t) = \frac{d\,x(t)}{d\,t} = \omega_n A\cos(\omega_n t + \phi)$$

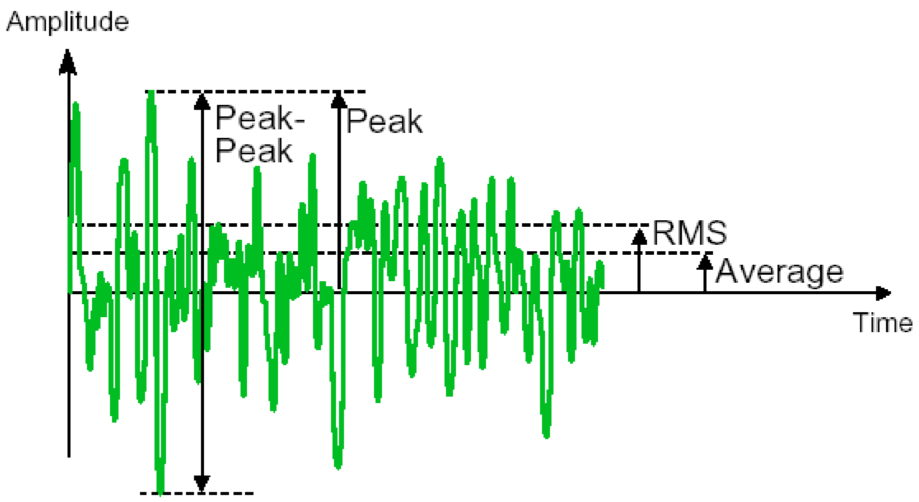
$$= \omega_n A\sin(\omega_n t + 90^\circ + \phi)$$

Acceleration

$$a(t) = \frac{d\,v(t)}{d\,t} = -\omega_n{}^2 A\sin(\omega_n t + \phi)$$

$$= \omega_n{}^2 A\sin(\omega_n t + 180^\circ + \phi)$$





Period : T

$$T = \frac{2\pi}{\omega}$$

Frequency: f

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

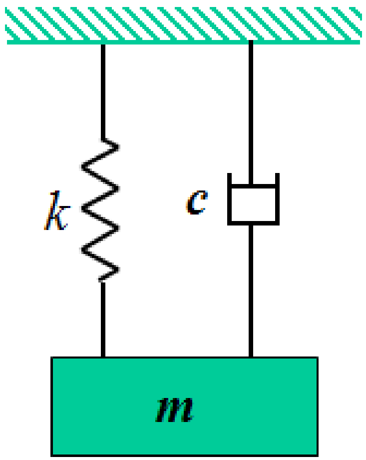
Average value

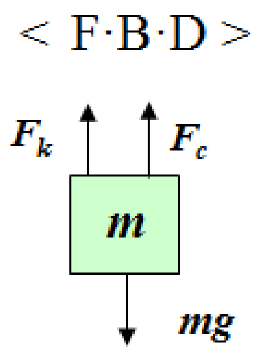
$$\bar{x} = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)\,dt$$

Root mean square value (RMS value)

$$RMS\left(x\right) = \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t)\,dt}$$

3. 1자유도 시스템의 감쇠계 자유진동





$$m\ddot{x} + c\dot{x} + kx = 0 \quad (1)$$

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

$$x = a e^{\lambda t} \quad (2)$$

(2) ➔(1)

$$\sum \vec{F} = m\vec{a}$$

$$-k(x+\Delta) - c\frac{d}{dt}(x+\Delta) + mg = m\ddot{x}$$

$$(m\lambda^2 + c\lambda + k)\, ae^{\lambda t} = 0$$

$$m\lambda^2 + c\lambda + k = 0$$

$$\therefore \ \lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

① critical damping

$$c_{cr} - 4mk = 0$$

$$\therefore c_{cr} = 2\sqrt{mk} \tag{4}$$

② damping ratio

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}} \tag{5}$$

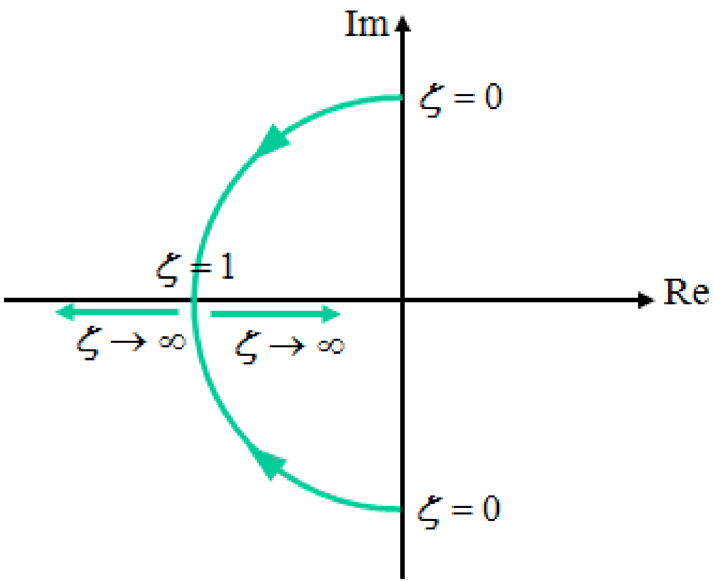
(5) ➔ (3)

$$\therefore \lambda_{1,2} = -\zeta\,\omega_n t \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$0 < \zeta < 1$$

$$\zeta = 1$$

$$\zeta > 1$$



Locus of $\lambda_{1,2}$ with the variation of damping ratio

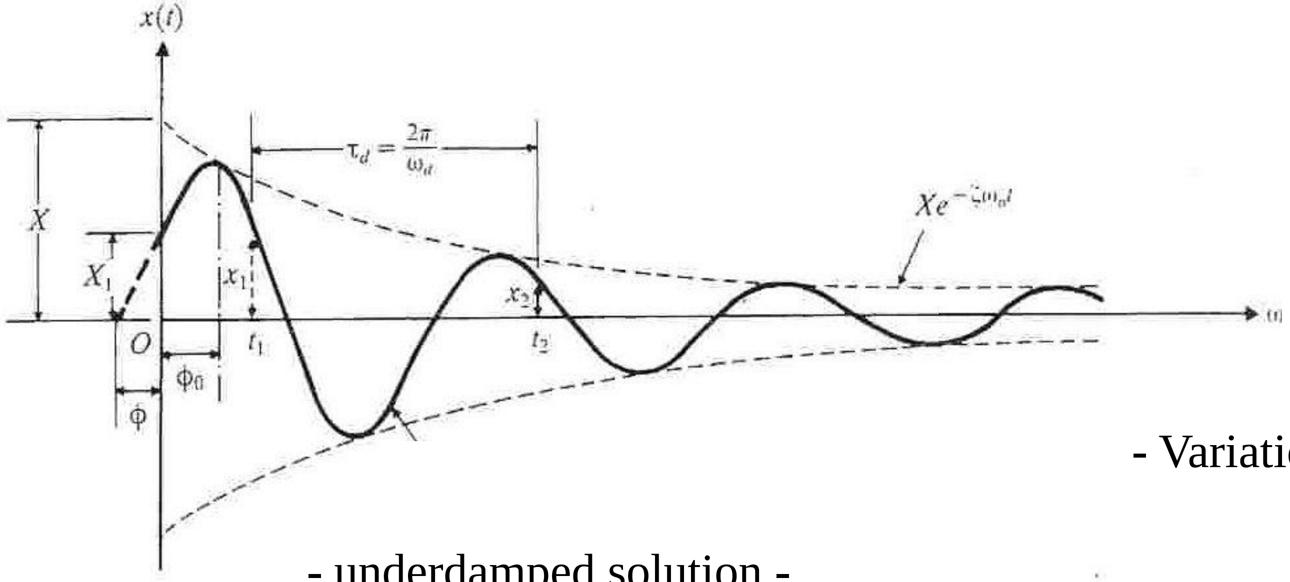
(A) underdamped system

$$\lambda_{1,2} = -\zeta\,\omega_n \pm j\,\omega_n\sqrt{1-\zeta^2}$$

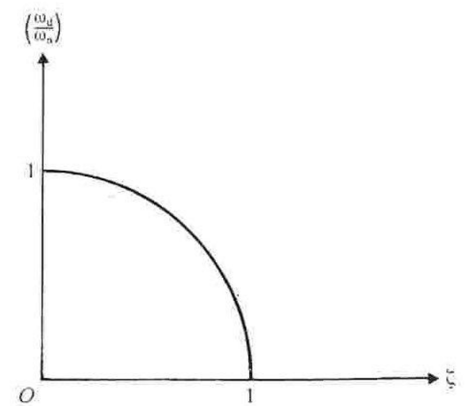
$$x(t) = e^{-\zeta\,\omega_n t}\left(a_1\, e^{j\omega_n\sqrt{1-\zeta^2}\,t} + a_2 e^{-j\omega_n\sqrt{1-\zeta^2}\,t}\right)$$

$$= X\,e^{-\zeta\,\omega_n t}\sin(\omega_d t + \phi)$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \quad [rad/s]$$



- underdamped solution -



- Variation of ω_d with damping ratio -

(B) critically damped system

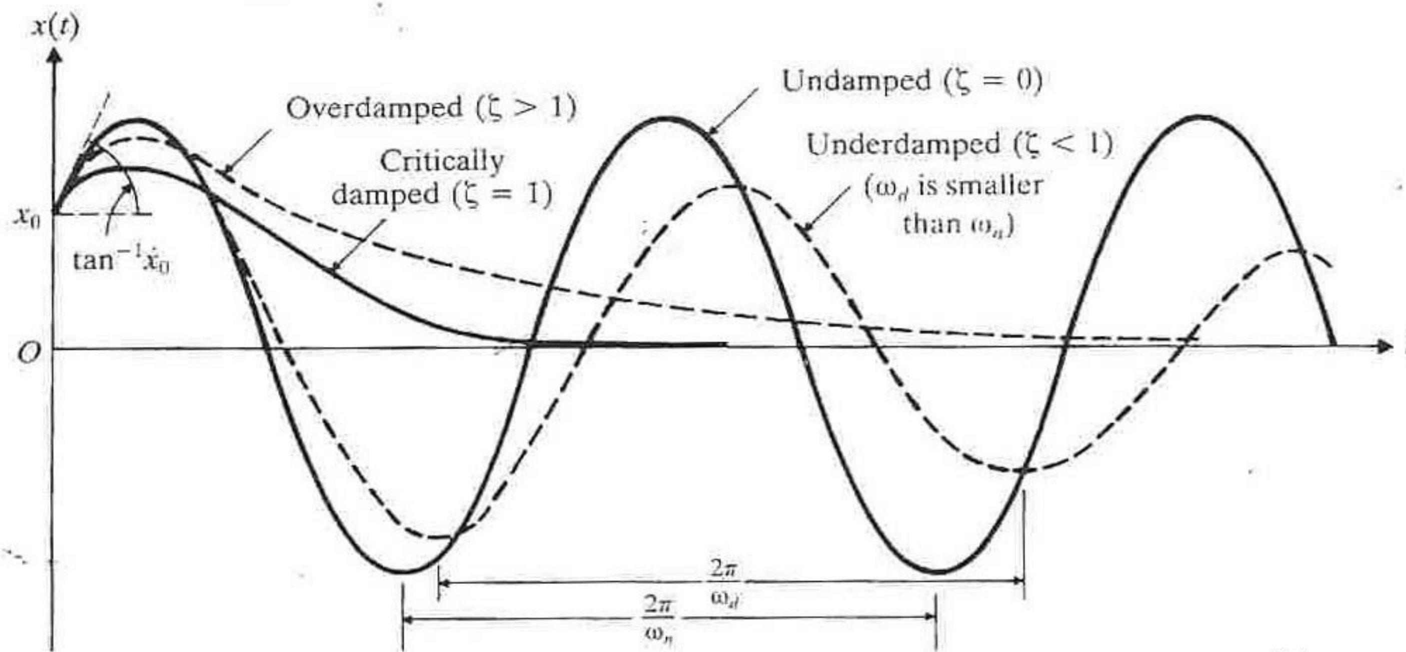
$$\lambda_{1,2} = -\omega_n$$

$$x(t) = (a_1 + a_2\, t)\, e^{-\omega_n t}$$

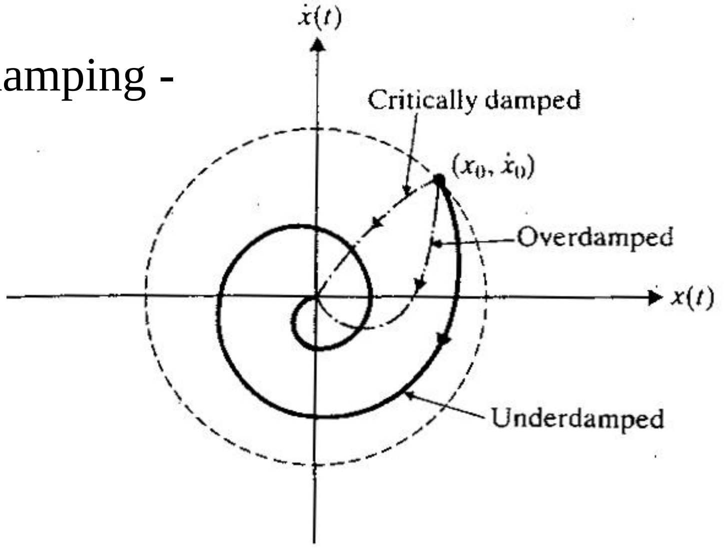
(C) overdamped system

$$\lambda_{1,2} = -\zeta\,\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

$$x(t) = e^{-\zeta\,\omega_n t} \left(a_1\, e^{\omega_n \sqrt{\zeta^2 - 1}\, t} + a_2 e^{-\omega_n \sqrt{\zeta^2 - 1}\, t} \right)$$



- Comparison of motions with different types of damping -



※ Energy dissipated in viscous damping

$$\frac{dW}{dt} = force \times velocity = F \cdot v$$

$$= (-cv) \cdot v$$

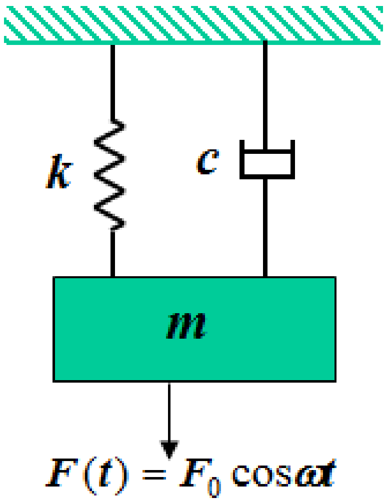
$$= -c\left(\frac{dx}{dt}\right)^2$$

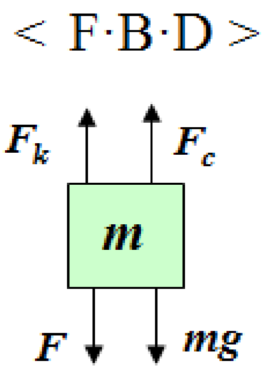
$$\Delta W = \int_0^{(2\pi/\omega_d)} c\left(\frac{dx}{dt}\right)^2 dt$$

$$= \int_0^{(2\pi/\omega_d)} c\left(\frac{d(X\sin\omega_d t)}{dt}\right)^2 dt$$

$$= \pi c \omega_d X^2$$

4. 1자유도 시스템의 감쇠계 강제진동





$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad (6)$$

$$x(0) = x_0, \quad \dot{x}(0) = v_0$$

$$x(t) = x_h(t) + x_p(t)$$

ω :

$$\sum \vec{F} = m\vec{a}$$

$$-k(x+\Delta) - c\frac{d}{dt}(x+\Delta) + mg + F(t) = m\ddot{x}$$

(A) Undamped case

(1) $\omega_n \neq \omega$

$$x(t) = A\sin\omega_n t + B\cos\omega_n t + \frac{F_0/m}{\omega_n^2 - \omega^2}\cos\omega t$$

$$= \frac{v_0}{\omega_n}\sin\omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2}\right)\cos\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

$$X = \frac{F_0/m}{\omega_n^2 - \omega^2}$$

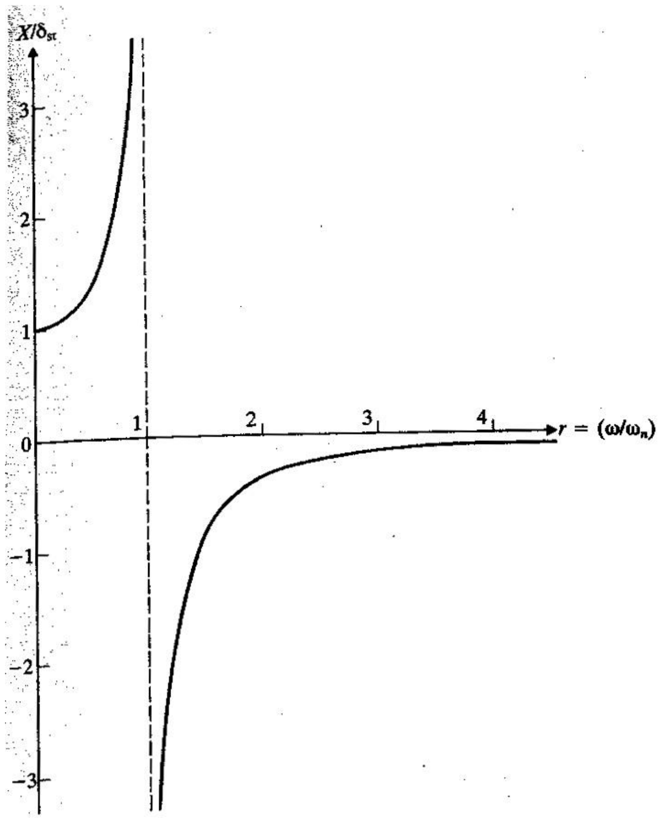
$$\delta_{st} = \frac{F_0}{k}$$

$$\kappa = \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

(2) $\omega_n = \omega$

$$x(t) = A_1 \sin\omega t + B_1 \cos\omega t + \frac{f_0}{2\omega} t \sin\omega t$$

$$= \frac{v_0}{\omega} \sin\omega t + x_0 \cos\omega t + \frac{f_0}{2\omega} t \sin\omega t$$

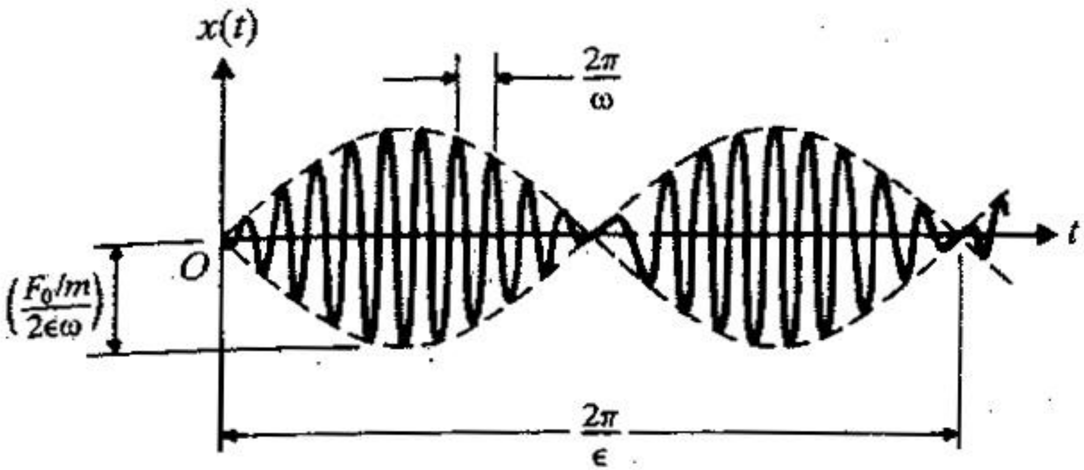


- Variation of magnification factor with frequency ration -

(3) $\omega_n \approx \omega$ (assume $x_0 = v_0 = 0$)

$$x(t) = \frac{f_0}{\omega_n^2 - \omega^2}(\cos\omega t - \cos\omega_n t)$$

$$= \frac{2f_0}{\omega_n^2 - \omega^2}\sin\left(\frac{\omega_n - \omega}{2}t\right)\sin\left(\frac{\omega_n + \omega}{2}t\right)$$



- Beating phenomenon -

Period of beating : τ_b

$$\tau_b = \frac{2\pi}{\omega_n - \omega}$$

Frequency of beating : ω_b

$$\omega_b = \omega_n - \omega$$

(B) Damped case

$$x_p(t) = X\cos(\omega t - \theta)$$

$$= \frac{f_0}{\sqrt{(\omega_n{}^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}\cos\left(\omega t - \tan^{-1}\frac{2\zeta\omega_n\omega}{\omega_n{}^2 - \omega^2}\right)$$

Where,

$$f_0 = \frac{F_0}{m}$$

In the steady state

Dynamic magnification factor

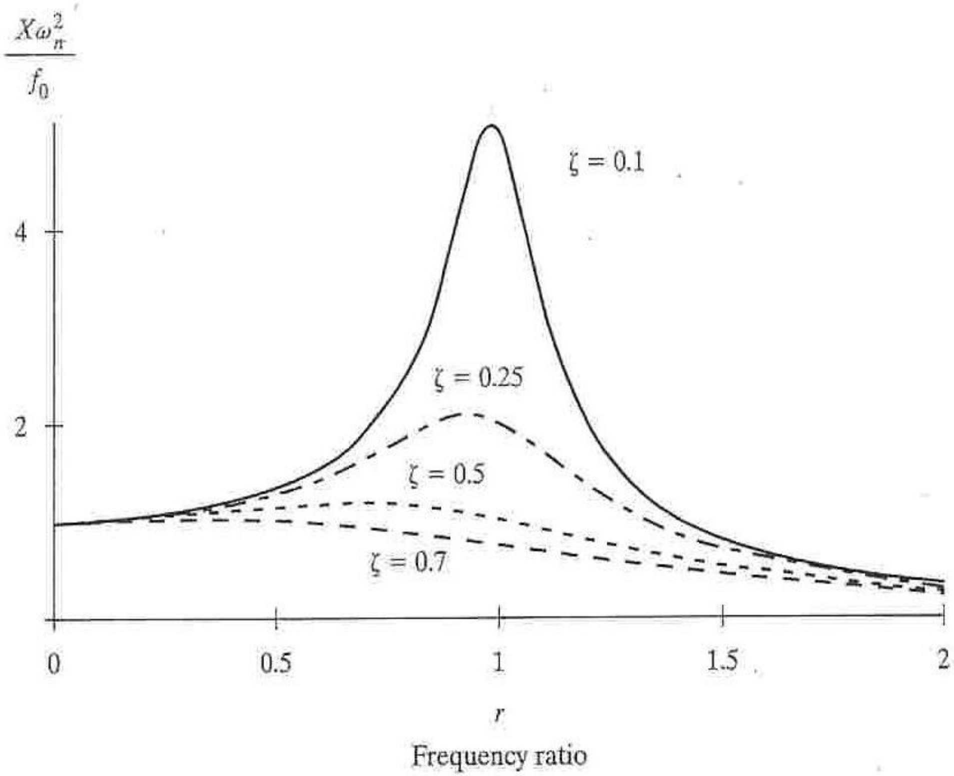
$$\kappa = \frac{X}{F_o/k} = \frac{X\,k}{F_0} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

F_0/k : static displacement

$$r = \frac{\omega}{\omega_n}$$

Phase

$$\theta = \tan^{-1}\frac{2\zeta r}{1-r^2}$$



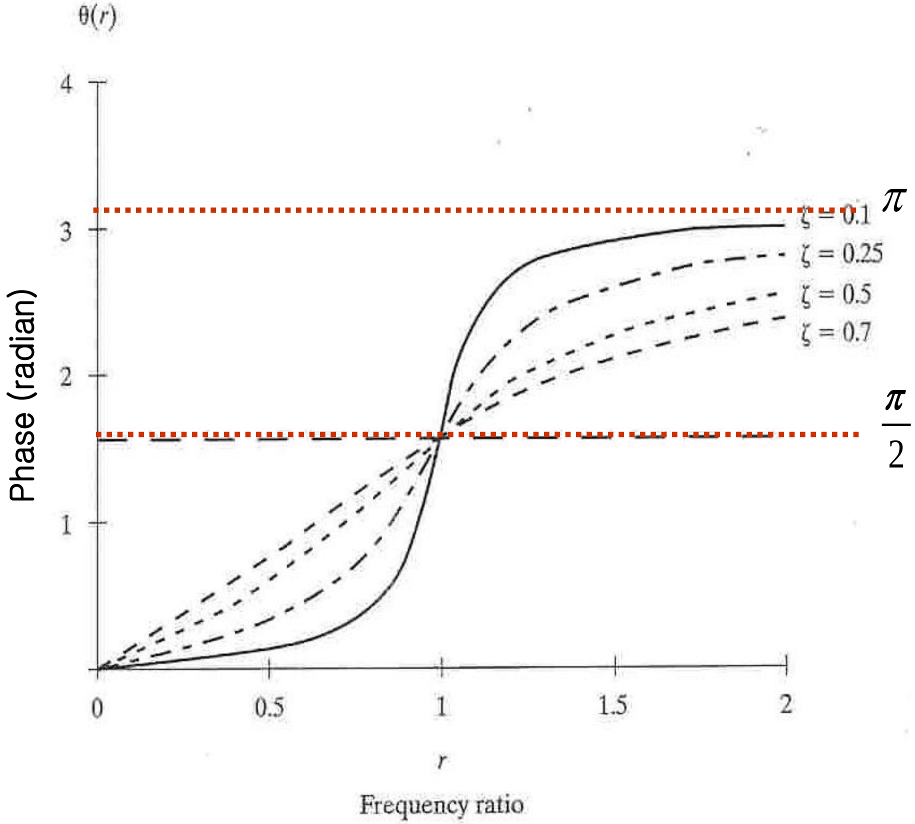
Peak magnitude of amplification factor

1) Case I $\left(0 \le \zeta \le \frac{1}{\sqrt{2}}\right)$

$$\kappa = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \text{at } \boldsymbol{r_{peak}} = \sqrt{1-2\boldsymbol{\zeta}^2}$$

2) Case II $\left(\zeta > \frac{1}{\sqrt{2}}\right)$

$$\kappa = 1 \qquad \text{at } r_{peak} = 0$$



1) Case I $(\zeta = 0)$

$(0 < r < 1)$

$(r > 1)$

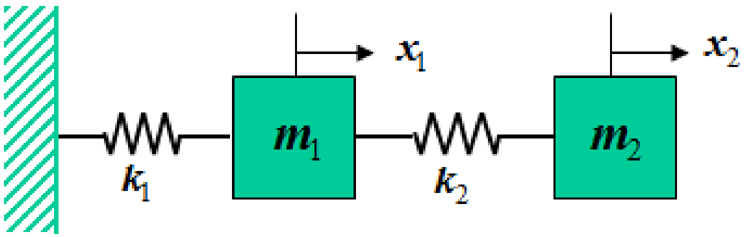
2) Case II $(\zeta > 0)$

$(0 < r < 1)$

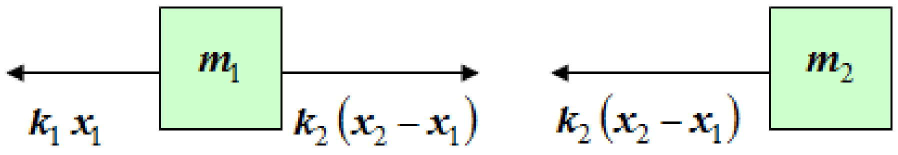
$(r = 1)$

$(r > 1)$

5. 다자유도 시스템의 비감쇠계 자유진동



< F·B·D >



$$\sum \vec{F} = m\vec{a}$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

$$x_1(0) = x_{10} \quad x_2(0) = x_{20}$$

$$\dot{x}_1(0) = \dot{x}_{10} \quad \dot{x}_2(0) = \dot{x}_{20}$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

$$\{x_0\} = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} \qquad \{\dot{x}_0\} = \begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} \tag{8}$$

Where,

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$\{x\} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

$$\{x(t)\} = \{u\} e^{j\omega t} \tag{9}$$

(9) ➔(8)

$$\left(-\omega^2 [M] + [K]\right)\{u\} e^{j\omega t} = \{0\}$$

For nontrivial solution

$$\det\left(-\omega^2[M]+[K]\right)=0$$

$$\det\begin{bmatrix} -\omega^2 m_1+k_1+k_2 & -k_2 \\ -k_2 & -\omega^2 m_2+k_2 \end{bmatrix}=0$$

$$m_1 m_2\omega^4-\left(m_1 k_2+m_2 k_1+m_2 k_2\right)\omega^2+k_1 k_2=0$$

① $\omega_1{}^2$ $\omega_2{}^2$

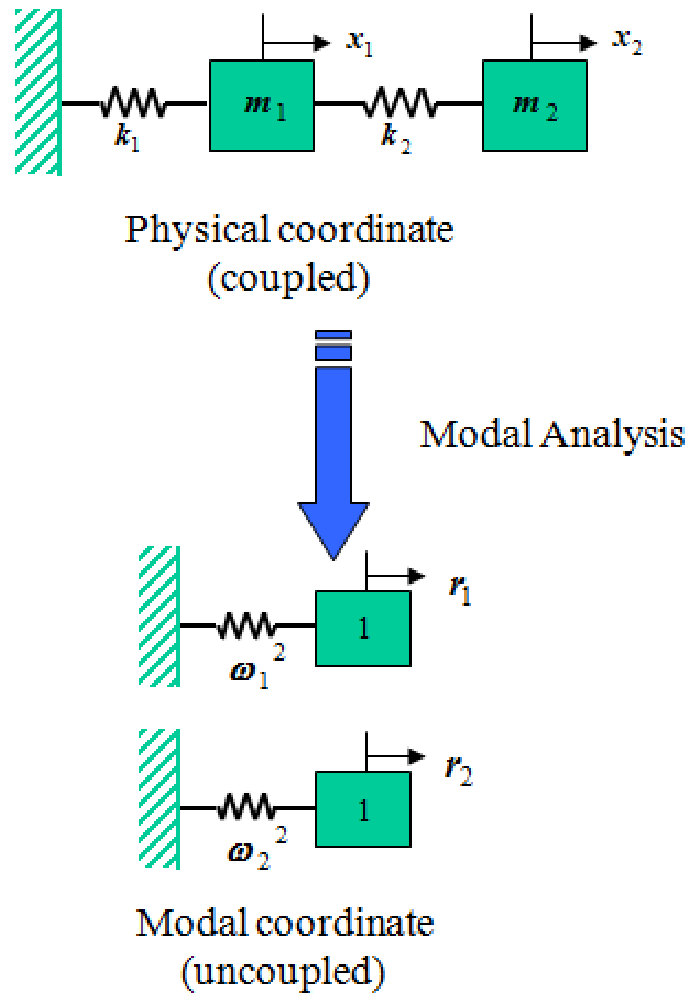
② $\left(-\omega_1{}^2[M]+[K]\right)\{u_1\}=\{0\}$

$\left(-\omega_2{}^2[M]+[K]\right)\{u_2\}=\{0\}$

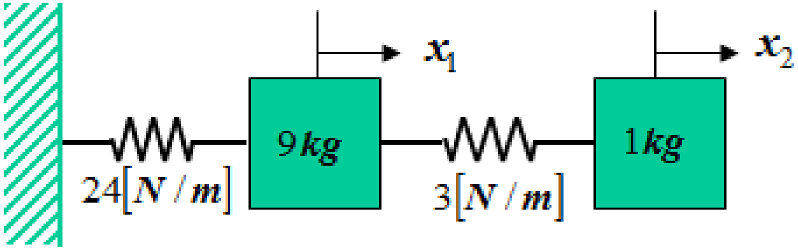
By superposition principle, the general solution of the system

$$\{\boldsymbol{x}(\boldsymbol{t})\}=\left(\boldsymbol{a}\boldsymbol{e}^{j\omega_1 t}+\boldsymbol{b}\boldsymbol{e}^{-j\omega_1 t}\right)\{\boldsymbol{u}_1\}+\left(\boldsymbol{c}\boldsymbol{e}^{j\omega_2 t}+\boldsymbol{d}\boldsymbol{e}^{-j\omega_2 t}\right)\{\boldsymbol{u}_2\}$$

$$=\boldsymbol{A}_1\sin(\boldsymbol{\omega}_1\boldsymbol{t}+\boldsymbol{\phi}_1)\{\boldsymbol{u}_1\}+\boldsymbol{A}_2\sin(\boldsymbol{\omega}_2\boldsymbol{t}+\boldsymbol{\phi}_2)\{\boldsymbol{u}_2\}$$


x1
x2
m1
m2
k1
k2
Physical coordinate
(coupled)
Modal Analysis
r1
1
ω1 2
r2
1
ω2 2
Modal coordinate
(uncoupled)

ex1)



$$x_{10} = 1[mm], \qquad x_{20} = 0[mm]$$
$$\dot{x}_{10} = \dot{x}_{20} = 0$$

sol)

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\{\ddot{x}\} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\{x\} = \{0\} \tag{9}$$

$$\{x(t)\} = \{u\}e^{j\omega t} \tag{10}$$

(10) ➔(9)

$$\left(-\omega^2[M] + [K]\right)\{u\}e^{j\omega t} = \{0\}$$

$$\det\left(-\omega^2[M]+[K]\right)$$
$$=\det\left(-\omega^2\begin{bmatrix}9 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}27 & -3\\ -3 & 3\end{bmatrix}\right)=0$$

$$\therefore \omega_1=\pm\sqrt{2}\ [rad/s] \qquad \omega_2=\pm 2[rad/s]$$

① $\omega_1=\pm\sqrt{2}$

$$\left(-2[M]+[K]\right)\{u\}=\{0\}$$

$$\begin{bmatrix}-18+27 & -3\\ -3 & -2+3\end{bmatrix}\begin{Bmatrix}u_{11}\\ u_{21}\end{Bmatrix}=\{0\}$$

$$\therefore \{u_1\}=k_1\begin{Bmatrix}1\\ 3\end{Bmatrix}$$

② $\omega_2 = \pm 2$

$$\left(-4[M] + [K]\right)\{u\} = \{0\}$$

$$\therefore \{u_2\} = k_2 \begin{Bmatrix} -1 \\ 3 \end{Bmatrix}$$

$$\therefore \{x(t)\} = A_1 \sin(\omega_1 t + \phi_1)\{u_1\} + A_2 \sin(\omega_2 t + \phi_2)\{u_2\}$$

From initial conditions

$$x_1(t) = \frac{1}{2}\sin\left(\sqrt{2}t + \frac{\pi}{2}\right) + \frac{1}{2}\sin\left(2t + \frac{\pi}{2}\right)$$

$$x_2(t) = \frac{3}{2}\sin\left(\sqrt{2}t + \frac{\pi}{2}\right) - \frac{3}{2}\sin\left(2t + \frac{\pi}{2}\right)$$

6. 연속계 (continuous system)

Discrete system	Continuous system
Finite DOF	
Ordinary differential equation ODE (t)	
Initial value problem	
Vector space (eigenvector)	

◆ Example : String

- Governing equation

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right]=\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2} \qquad 0 < x < L \qquad (10)$$

- boundary conditions (fixed-fixed conditions)

$$y(0,t)=y(L,t)=0$$

Sol)

By separation of variables

$$y(x,t)=Y(x)F(t) \qquad (11)$$

(11) ➔(10), $\rho(x)$, $T(x)$ = constant

$$\frac{1}{\rho Y(x)}\frac{d}{dx}\left[T\frac{dY(x)}{dx}\right]=\frac{1}{F(t)}\frac{d^2F(t)}{dt^2} \qquad (12)$$

$$\begin{cases} \dfrac{d^2 F(t)}{d t^2} + \omega^2 F(t) = 0 & (13) \\ -\dfrac{d}{d x}\left[T \dfrac{d Y(x)}{d x}\right] = \omega^2 \rho Y(x) \qquad 0 < x < L & (14) \\ Y(0) = Y(L) = 0 \end{cases}$$

From eq (14)

$$Y(x) = A \sin \beta x + B \cos \beta x \qquad \beta = \omega \sqrt{\frac{\rho}{T}}$$

$$Y(0) = B = 0$$

$$Y(L) = A \sin \beta L = 0$$

For non-trivial solution $\sin \beta L = 0$

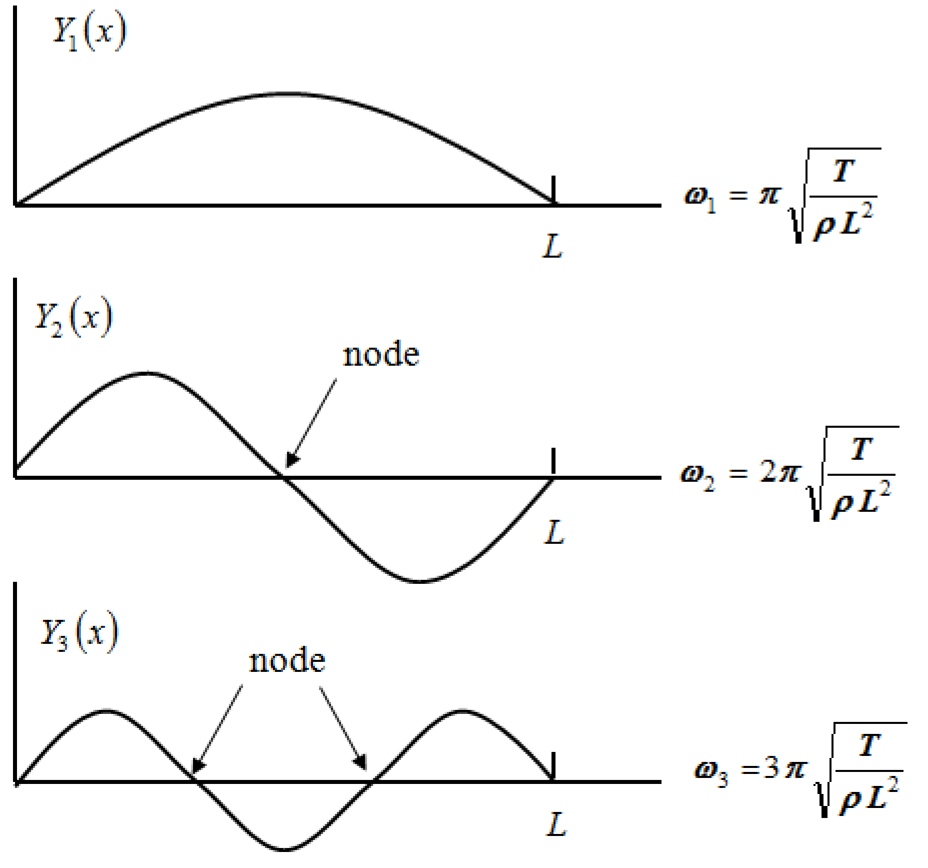
$$\beta_r L = r \pi \qquad r = 1, 2, 3, \cdots$$

Natural frequency

$$\omega_r = r\pi\sqrt{\frac{T}{\rho L^2}} \qquad r = 1, 2, 3, \cdots$$

Eigenfunction

$$Y_r = A_r \sin r\pi \frac{x}{L} \qquad r = 1, 2, 3, \cdots$$

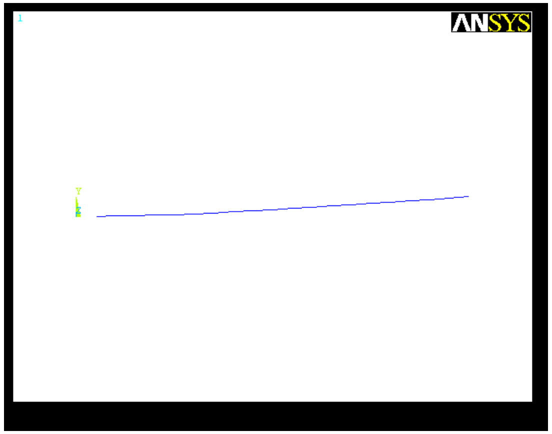


- Eigenfunctions of string -

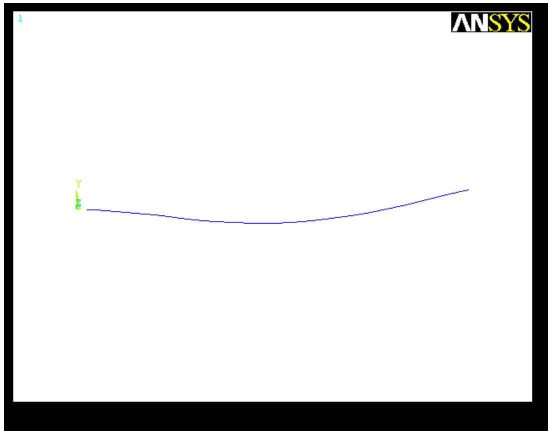
※ 유한요소 해석

◆ Example : Beam

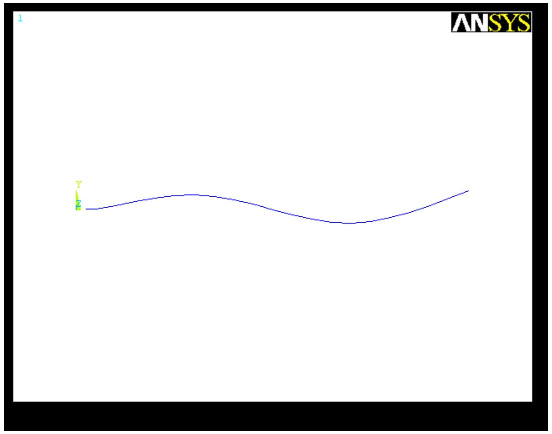
- First mode



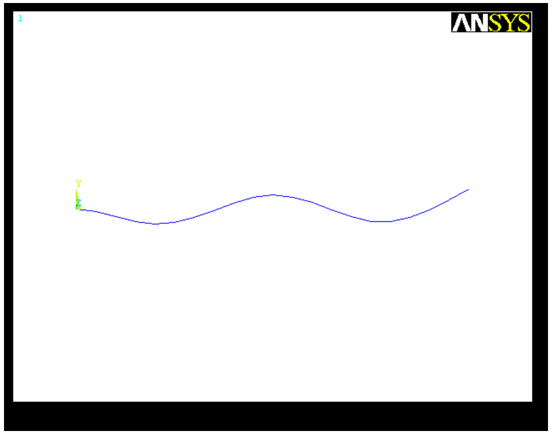
- Second mode



- Third mode

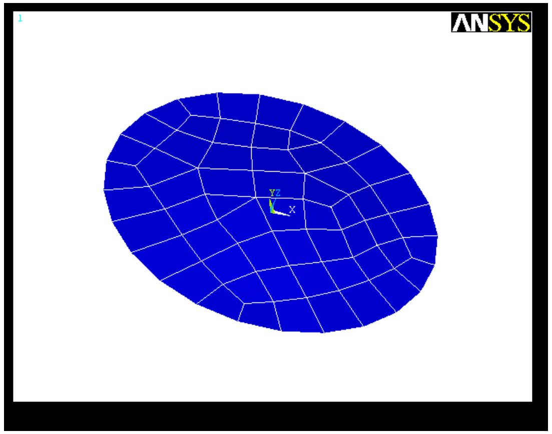


- Forth mode

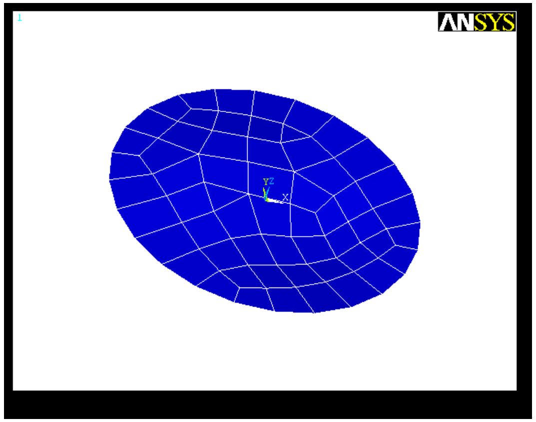


♦ Example : Disk

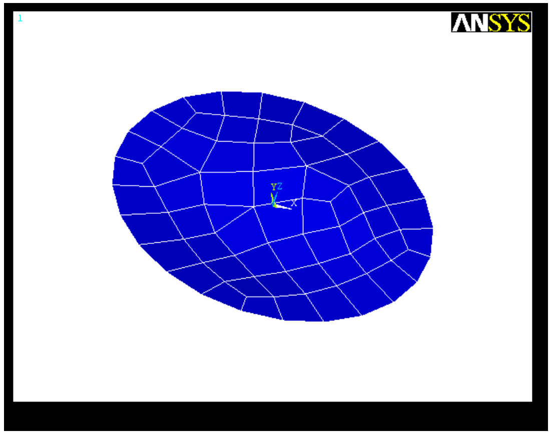
- First mode



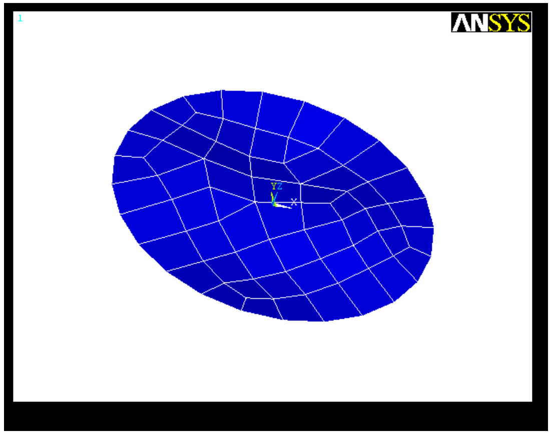
- Second mode



- Third mode

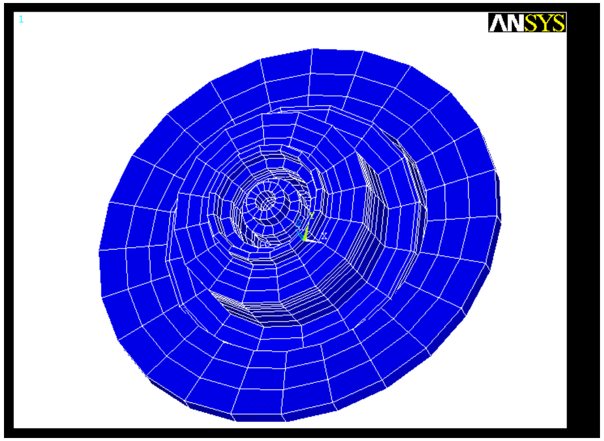


- Forth mode

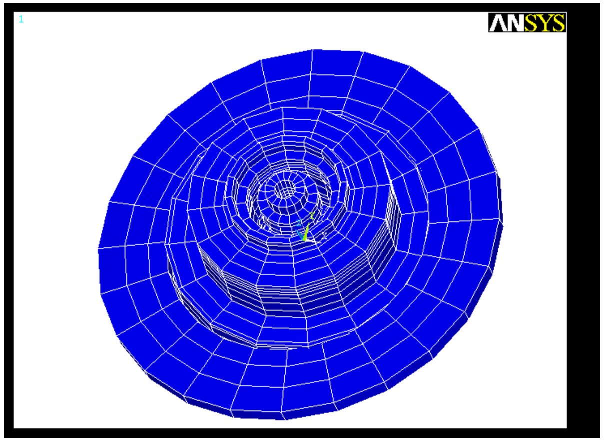


♦ Example : HDD

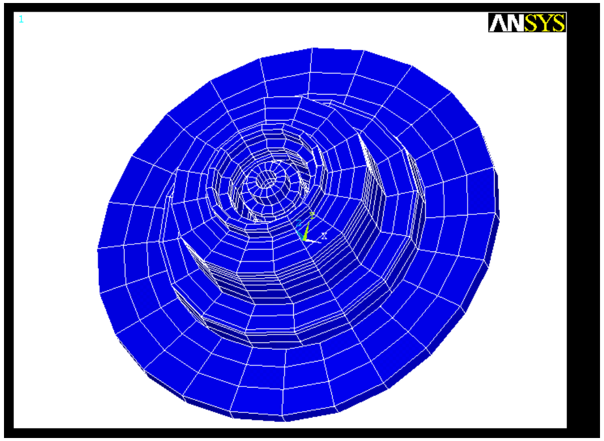
- First mode



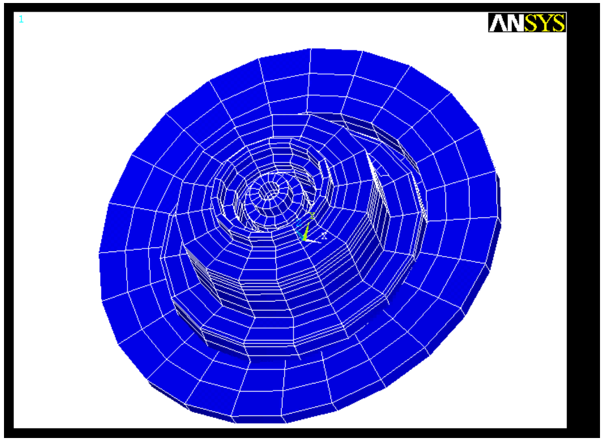
- Second mode



- Third mode

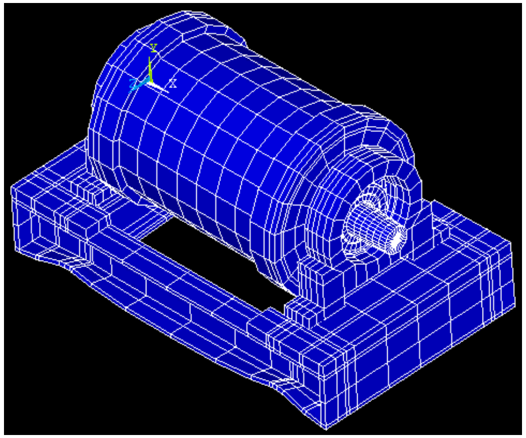


- Forth mode

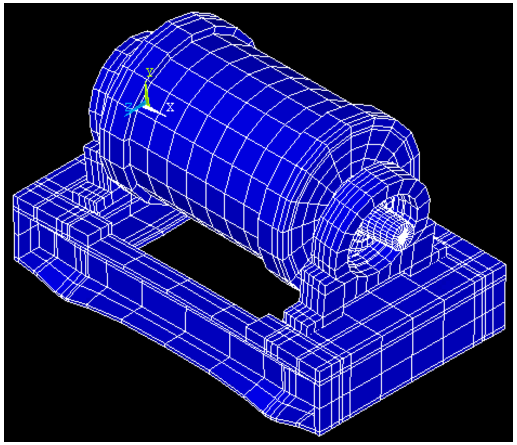


♦ Example : Dynomometer

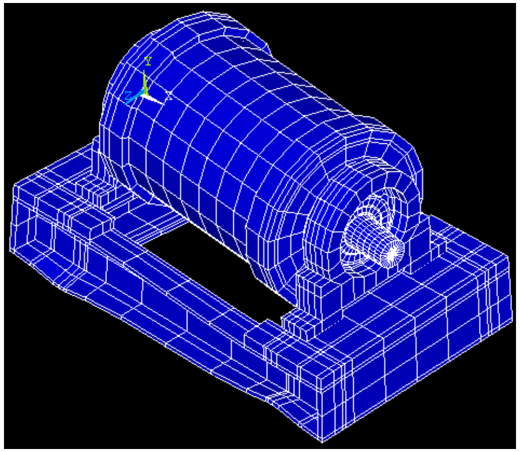
- First mode



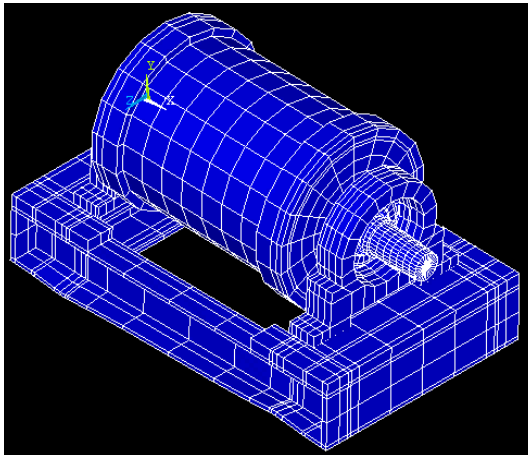
- Second mode



- Third mode

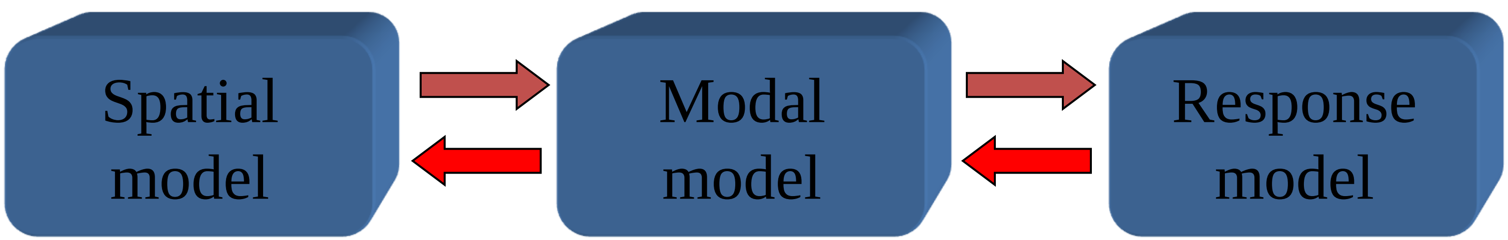


- Forth mode



7. 모달테스팅 (modal testing)

: processes involved in testing components or structures with the objective of obtaining a mathematical description of their dynamic or vibration behavior



Description of structure
- mass, spring, damper

Vibration mode
- natural frequencies
- mode shapes
- modal damping

Response level
- Frequency response function

(1) Basic measurement system

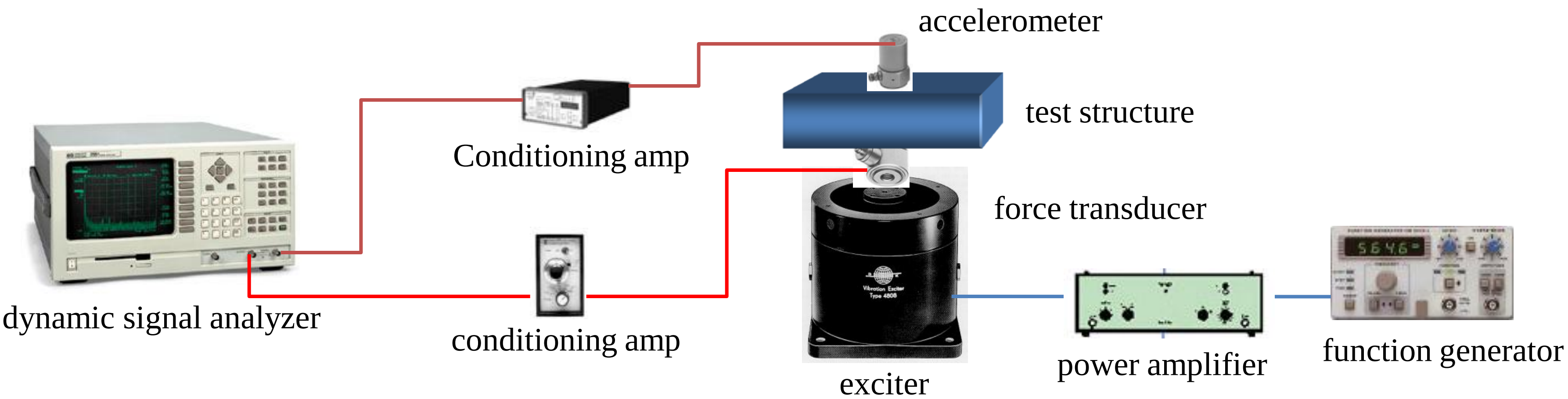
(1) Excitation system

(2) Transduction system

(3) Amplifier

(4) Analyzer(Dynamic signal analyzer)

(A) Shaker test



(B) Impact test